

Distribution & Interpolation Spaces – Exercise sheet 2

Exercise 1. Let $E \subset \mathbb{R}^n$ be a measurable set and $f : E \rightarrow [0, +\infty]$ be a measurable function. Prove that

$$\int_E f(x) dx = \int_0^\infty |\{x \in E : f(x) > t\}| dt,$$

where $|A|$ denotes the Lebesgue measure of the set A . Let $p \in (1, +\infty)$ and prove that

$$\int_E f(x)^p dx = p \int_0^\infty t^{p-1} |\{x \in E : f(x) > t\}| dt.$$

Exercise 2. Let $f \in L^1(\mathbb{R}^n)$ be such that

$$\int_{\mathbb{R}^n} f\varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

Show that $f = 0$ almost everywhere on \mathbb{R}^n .

Exercise 3. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $(K_m)_m$ be a sequence of compact sets for which the following holds

$$K_m \subseteq K_{m+1} \quad \forall m \in \mathbb{N} \quad \text{and} \quad \Omega = \bigcup_{m \in \mathbb{N}} K_m,$$

and K_1 is assumed to be the empty set. We denote with $\|\cdot\|_{m,k} : C^\infty(\Omega) \rightarrow [0, \infty)$ the functions

$$\|\varphi\|_{m,k} = \max\{|D^\alpha \varphi(x)| : |\alpha| \leq k, x \in K_{m+1} \setminus K_m\},$$

namely the maximal value among the norms of *sup* of all derivatives of φ up to order k in $K_{m+1} \setminus K_m$. Show that $\|\cdot\|_{m,k}$ is a semi-norm on $C_c^\infty(\Omega)$.

Exercise 4. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $\varphi \in C_c^\infty(\Omega)$ be a function and

$$\mathbb{N} \ni m \mapsto k(m) \in \mathbb{N} \quad \text{and} \quad \mathbb{N} \ni m \mapsto \lambda(m) \in \mathbb{R}_{\geq 0}$$

be two sequence of respectively integers and positive numbers. Show that the following collection of sets

$$\mathcal{D}_{\varphi,k,\lambda} = \{\psi \in C_c^\infty(\Omega) : \forall m \in \mathbb{N}, \|\varphi - \psi\|_{m,k(m)} < \lambda(m)\}$$

for different choices of $\varphi, k(m), \lambda(m)$, is a basis generating a topology on $C_c^\infty(\Omega)$.

Exercise 5. Show that the topology introduced in Exercise 4 induces the following notion of convergence in $C_c^\infty(\Omega)$.

A sequence $(\varphi_k)_k \subset C_c^\infty(\Omega)$ converges to $\varphi \in C_c^\infty(\Omega)$ if and only if there exists a compact set $K \subset \Omega$ such that $\text{spt } \varphi_k \subseteq K$ for all $k \in \mathbb{N}$ and all φ_k and their derivatives of any order converge uniformly to φ and its respective derivative.

Definition A map $f : X \rightarrow Y$ between two topological spaces is continuous at $x \in X$ if for any neighbourhood U of $f(x)$ there is a neighbourhood V of x such that $f(V) \subset U$.

Exercise 6. A map $f : X \rightarrow Y$ is continuous if and only if it is continuous at every point of X .

From the previous property deduce the Theorem 1.1.11 stated in class.